CALCULUS II
THE HYPERBOLIC ARGUMENT

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We may parameterize the circle $x^2 + y^2 = 1$ with the path $\gamma(t) = (\cos t, \sin t)$. Viewing t as time, we see that the arclength of the curve traced by γ is exactly the radian measure of the angle which has been traced. The area A of the sector swept out in time t is $(\frac{t}{2\pi})\pi r^2 = \frac{t}{2}$. Thus we may interpret t not only as the angle produced by $\gamma(t)$, but also as $2A$, twice the area swept out in time t.

We investigate an analogous interpretation for the hyperbola given by the hyperbolic functions.

We may parameterize the hyperbola $x^2 - y^2 = 1$ with the path $\gamma(t) = (\cosh t, \sinh t)$. In the case of the circle, t is an angle, and $2t$ is the area swept out. Although the angle does not carry over the the hyperbolic case, the area interpretation does, as we wish to show.

Let $A(t)$ be the area between the x-axis, the line through the origin and the point $\gamma(t)$, and the curve $x^2 - y^2 = 1$. Then $A(t)$ is the area of the triangle with vertices $(0,0)$, $(\cosh t, 0)$, and $(\cos t, \sinh t)$, minus the area under the curve $y = \sqrt{x^2 - 1}$. Thus

$$A(t) = \frac{1}{2}\cosh t \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\, dx.$$

We wish to compute the derivative of $A(t)$.

Let $G(t) = \cosh t \sinh t$. Then by the product rule,

$$\frac{dG}{dt} = \cosh t(\cosh t) + \sinh t(\sinh t) = \cosh^2 t + \sinh^2 t = 1 + 2\sinh^2 t.$$

Let $F(t) = \int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$. We use the Fundamental Theorem of Calculus to compute the derivative $\frac{dF}{dt}$. Let $u = \cosh t$, and note that $\frac{du}{dt} = \sinh t$. Now $F(u) = \int_1^u \sqrt{x^2 - 1}\, dx$, so by FTC, $\frac{dF}{du} = \sqrt{u^2 - 1} - \sqrt{1 - 1} = \sqrt{u^2 - 1} = \sqrt{\cosh^2 t - 1} = \sinh t$. Then

$$\frac{dF}{dt} = \frac{dF}{du}\frac{du}{dt} = \sinh^2 t.$$

Computing the derivative of the area function above gives

$$\frac{dA}{dt} = \frac{1}{2}\frac{dG}{dt} - \frac{dF}{dt} = \frac{1}{2}(1 + 2\sinh^2 t) - \sinh^2 t = \frac{1}{2}.$$

Thus

$$A(t) = \int_0^t \frac{1}{2}\, dx = x \,|_0^t = \frac{t}{2}.$$

So we may interpret t as $2A$, twice the area swept out by γ in time t.

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